Algebraic structures in exactly solvable models (towards higher-dimensional integrability?)

PhD project

Supervision

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Keywords

This project lies at the interface between mathematics and theoretical physics. It is a melting pot of algebra, representation theory, functional analysis and statistical mechanics, aiming at studying the mathematical structures present in the exactly solvable ("integrable") models of statistical mechanics, with potential applications to problems in classical or quantum many-body physics.

Scientific context

Exactly solvable models play an essential role in our understanding of the emergence of macroscopic laws from microscopic ones, for instance in the study of phase transitions. Those are models where certain physical quantities can be computed analytically in the thermodynamic limit, whereas numerical approaches typically face severe difficulties due to the large number of constituents. Applications include, to name a few: classical magnetism [1], two-dimensional geometrical models such as percolation [3] or loop models [4], stochastic models [5] or one-dimensional quantum systems [6] (in the two latter cases, time plays the role of the extra dimension, such that these can generally be related to two-dimensional statistical mechanical models at equilibrium).

The two seminal works in the field of exactly solvable models are Bethe's diagonalization of the Heisenberg spin chain, a one-dimensional model of interacting quantum magnets (1931) [2], and Onsager's solution of the two-dimensional Ising model (1944) [1], a typical configuration of which is shown on the left panel of Fig. 1. Bethe's work later gave rise to a beautiful field of research known as *quantum integrability*, deeply rooted in algebraic structures such as the *Yang-Baxter equation* and *quantum groups*. The Yang-Baxter equation is a consistency relation satisfied by the local statistical weights of a model, and can be viewed as the primary ingredient for commuting transfer matrices and exact solvability. Quantum groups (typically denoted $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra and q a deformation parameter), in turn, are quasitriangular Hopf algebras which provide the mathematical framework for the Yang-Baxter equation. Another deeply related object is the Temperley-Lieb algebra (see right panel of Fig. 1).

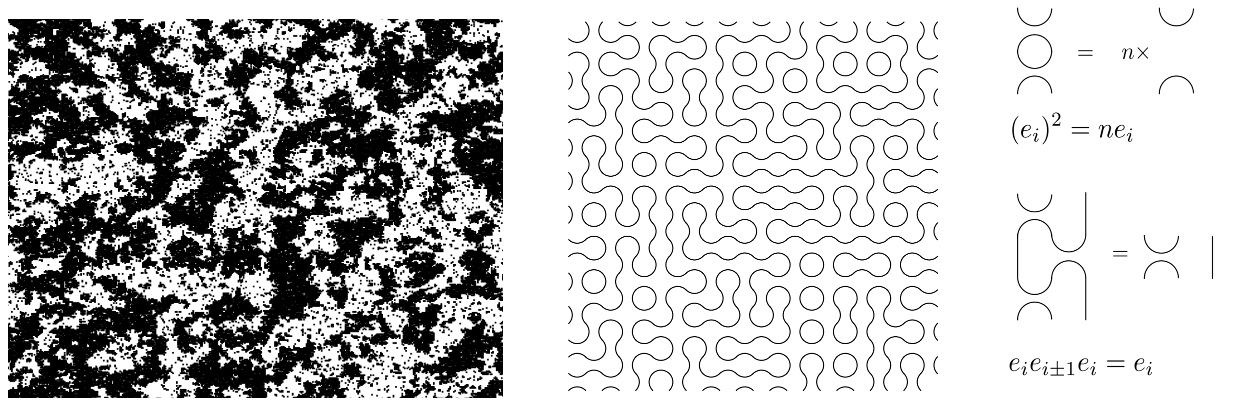

Figure 1: Left: Two-dimensional Ising model at its critical point. Middle: random configuration of two-dimensional fully packed loops. Right: Graphical depiction of the Temperley-Lieb algebra, which encodes the configurations of the two-dimensional loop model into algebraic rules obeyed by generators $\{e_i\}$.

The techniques of quantum integrability have allowed for exact solutions in a variety of problems in two-dimensional statistical mechanics as well as for 1+1-dimensional quantum systems. Notably, however, most exact solutions so far have remained limited to low-dimensional systems, a fact which can be interpreted as a consequence of the "rigidity" of the algebraic objects at play. There have been attempts to promote the Yang-Baxter equation to a three-dimensional *tetrahedron equation* [7], however all solutions of the latter found this far fail to relate to genuine three-dimensional physical models.

Overview of the project

The objective of this project is to revisit the algebraic structures of integrable models under a different light, which could provide another route towards an extension to higher dimension, in addition to bringing new insight for low-dimensional problems.

Our starting point will be the study of integrable models at the special points where the quantum group deformation parameter q is a root of unity. On the one hand those cover many of the physically relevant cases (Ising model, critical Potts models [10], percolation [11], restricted solid-on-solid models [9], dilute or dense polymers [12, 13], etc...). On the other hand, at such points the underlying algebraic structures are considerably enriched [14]: while the representation theory of a quantum group $U_q(\mathfrak{g})$ at generic q essentially reproduces that of the universal enveloping algebra $U(\mathfrak{g})$ (in particular all finite-dimensional representations become decomposable), at root of unity even finite-dimensional representations admit non-trivial extensions, and the representation theory of $U_q(\mathfrak{g})$ reminds the one of simple Lie groups at positive characteristic [8]. This comes with important physical consequences, ranging from the relation with Logarithmic Conformal Field Theory [15] to the relaxation properties of many-body quantum systems [16].

In a relatively recent work [17], one of the advisors (EV) has shown how at such root of unity points, traditional integrable models show a surprising connection with the Onsager algebra, an infinite-dimensional Lie algebra which played a crucial role in Onsager's historical solution of the Ising model but has since then remained quite remote from the whole development of quantum integrability. In this project we plan to explore further the different aspects of this connection, in particular:

- translate the Onsager algebra into quadratic relations between transfer matrices, which can prove a powerfool tool towards exact solvability.

- explore the case with open boundaries, where a past study by the other advisor (AG) has evidenced the appearance of yet another infinite-dimensional algebra, cousin to the Onsager algebra but which remains to be explored for the most part [18]

Depending on the taste of the candidate, we may also explore applications to quantum many-body systems. There, the Onsager algebra or its variants shows up at root of unity as a non-abelian symmetry algebra, which puts important constraints on the physics, for instance the non-equilibrium relaxation properties: while it is accepted that physical systems relax to equilibrium by maximizing their entropy under constraints fixed by their conservation laws, what happens when the conserved quantities do not commute with one another? Some aspects were studied in [19, 20], but many interesting facets remain to explore. In this respect, characterizing non-abelian symmetries algebra in terms of transfer matrix relations as mentioned above may prove a very useful tool to construct *non abelian Generalized Gibbs Ensembles*.

We will then step up to the three-dimensional case, where our hope is that some of the aspects described above may be carried over more naturally than the other algebraic structures traditionally attached to integrability. In particular, an interesting aspect of the work [17] is that the Onsager algebra was shown to emerge from two very simple conditions, namely, some $U(1)$ conservation law (number of particles conservation) together with self-duality. More precisely, a pair of generators A_0 and A_1, corresponding respectively to the "particle number" and its dual, were very naturally shown to obey a set of relations known as *Dolan Grady* [21], which in turn guarantees that they should generate a representation of the Onsager algebra under iterated commutations. Our strategy will therefore be to seek a three-dimensional analog of this construction: can we construct simple models with some elementary conserved quantity A_0 and invariance under some duality transformation $\mathcal{D}$? What algebraic relations do A_0 and $\mathcal{D}(A_0)$ obey? What is the algebra encoded in those relations?

A complementary direction we may follow regards the extension in three dimensions of two-dimensional geometrical lattice models such as loop models (see Fig. 1), and more generally the web models developped by one of the advisors (AG) in recent works [22]. While loop models are widely studied both in relation with integrability, algebra (with a prominent role played by the Temperley-Lieb algebra, see right panel of Fig. 1), conformal field theory and probability theory, web models are very much less so. These generalize the former by allowing bifurcation and branching (algebraically, their construction in 2D relies on objects known as *spiders* [23]) and are expected to play a role in the description of interfaces in spin models where the spins can take more than two values. In order to step up to three dimensions, we need define a statistical model of 3D cobordisms between spider configurations (also known as *foams* [24]) which at every generic time slice yields a configuration of the 2D web (or loop) model. On the algebraic side, these cobordisms are central tools in the construction of Khovanov homology, a knot homology theory that upgrades the Jones polynomial. This construction was introduced by Khovanov in the early 2000's [25], and has since been at the center of intensive research and extensions [26].

The work will involve both analytical and numerical parts. The candidate should have a background in mathematics and/or theoretical physics, and depending on their taste and the work's advancement, we may lean more towards one or the other discipline along the flow of the project.

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